CS 70Discrete Mathematics and Probability TheoryFall 2024Rao, HugDIS 3A

1 Graph Theory Basics

Note 5

For each type of graph, select all properties that are satisfied by the graph type.

- (a) Planar graph. Let v, e, f be the number of vertices, edges, and faces, respectively.
 - i. v + f = e + 2
 - ii. $e \leq 2v 4$
 - iii. Any planar graph can be vertex-colored in at most 5 colors.

(b) Tree

- i. Has |V| 1 edges
- ii. Can have cycles
- iii. Adding an edge will decrease the number of connected components
- (c) Hypercube. Let *n* be the dimension of the hypercube.
 - i. $|V| = 2^n$ ii. $|E| = n \cdot 2^{n-1}$

2 Always, Sometimes, or Never

Note 5 In each part below, you are given some information about a graph *G*. Using only the information in the current part, say whether *G* will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

- (a) G can be vertex-colored with 4 colors.
- (b) G requires 7 colors to be vertex-colored.
- (c) $e \le 3v 6$, where *e* is the number of edges of *G* and *v* is the number of vertices of *G*.
- (d) G is connected, and each vertex in G has degree at most 2.

(e) Each vertex in *G* has degree at most 2.

3 Graph Coloring

Note 5 Prove that a graph with maximum degree at most k is (k+1)-colorable.

4 Hypercubes

Note 5

The vertex set of the *n*-dimensional hypercube G = (V, E) is given by $V = \{0, 1\}^n$ (recall that $\{0, 1\}^n$ denotes the set of all *n*-bit strings). There is an edge between two vertices *x* and *y* if and only if *x* and *y* differ in exactly one bit position.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that the edges of an n-dimensional hypercube can be colored using n colors so that no pair of edges sharing a common vertex have the same color.

(c) Show that for any $n \ge 1$, the *n*-dimensional hypercube is bipartite.