Counting Intro II

Inclusion-exclusion: With two sets,

$$|A| \quad + \quad |B| \quad - \quad |A \cap B| \quad = \quad |A \cup B|$$
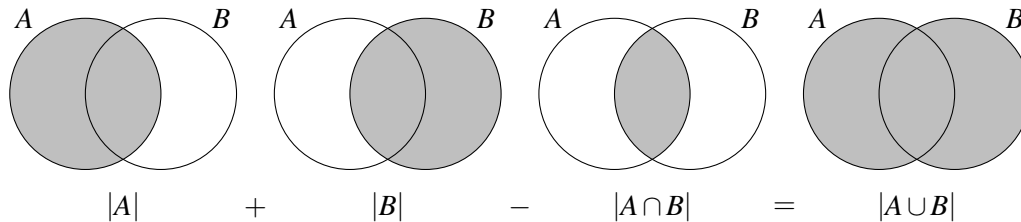
With more sets,

$$
\begin{aligned}
|A_1 \cup A_2 \cup \cdots \cup A_n| = \ & |A_1| + |A_2| + \cdots + |A_n| \\
& - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_i \cap A_j| - \cdots - |A_{n-1} \cap A_n| \\
& + |A_1 \cap A_2 \cap A_3| + \cdots + |A_i \cap A_j \cap A_k| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| \\
& \cdots
\end{aligned}
$$

$$
\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S|=k}} \left| \bigcap_{i \in S} A_i \right|
$$

That is, for each size k, iterate through all ways of picking k sets from $\{A_1, \ldots, A_n\}$, and alternate between adding and subtracting the sizes of their intersection.

Combinatorial proofs: A technique for proving combinatorial identities. There should be very little math involved (usually none): use two different ways of counting the same scenario. One way should correspond to the left-hand side of the equality, and the other way should correspond to the right-hand side of the equality. The fact that we're counting the same scenario means that the two sides are equal.

1 Inclusion and Exclusion

Note 10

What is the total number of positive integers strictly less than 100 that are also coprime to 100?

2 CS70: The Musical

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(b) Edward would now like to select a crew out of n people. Use this to provide a combinatorial argument that proves the following identity: (this is called Pascal's Identity)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity:

$$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}.$$

3 Countability Intro

Countability: Formal notion of different kinds of infinities.

- *Countable*: able to enumerate in a list (possibly finite, possibly infinite)

- *Countably infinite*: able to enumerate in an infinite list; that is, there is a bijection with $\mathbb{N}$.

  To show that there is a bijection, the *Cantor–Bernstein theorem* says that it is sufficient to find two injections, $f : S \to \mathbb{N}$ and $g : \mathbb{N} \to S$. Intuitively, this is because an injection $f : S \to \mathbb{N}$ means $|S| \leq |\mathbb{N}|$, and an injection $g : \mathbb{N} \to S$ means $|\mathbb{N}| \leq |S|$; together, we have $|\mathbb{N}| = |S|$.

- *Uncountably infinite*: unable to be listed out

  Use *Cantor diagonalization* to prove uncountability through contradiction; the classic example is the set of reals in $[0,1]$:

$\mathbb{N}$	$[0,1]$
0	0 . ⑦ 3 2 0 5 0 $\cdots$
1	0 . 4 ① 4 2 1 3 $\cdots$
2	0 . 6 1 ⑧ 0 3 3 $\cdots$
3	0 . 1 8 2 ⑧ 1 8 $\cdots$
4	0 . 1 4 1 5 ⑨ 2 $\cdots$
5	0 . 5 7 7 2 1 ⑤ $\cdots$
$\vdots$	$\vdots$
?	0 . 8 2 9 9 1 6 $\cdots$

  If we change the digits along the diagonal, the new decimal created is different from every single element in the list in at least one place, so it's not in the list—this is a contradiction.

Sometimes it can be easier to prove countability/uncountability through bijections with other countable/uncountable sets respectively. Common countable sets include $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N} \times \mathbb{N}$, finite length bitstrings, etc. Common uncountable sets include $[0,1]$, $\mathbb{R}$, infinite length bitstrings, etc.

(a) Your friend is confused about how Cantor diagonalization doesn't apply to the set of natural numbers. They argue that natural numbers can be thought of as an infinite length string of digits, by padding each number with an infinite number of zeroes to the left. If we then assume by contradiction that we can list out the set of natural numbers with the padded 0's, we can change the digits along a diagonal, to create a new natural number not in the list.

$$
\begin{array}{c|c}
0 & \cdots\ 0\ 0\ 0\ 0\ ①\\
1 & \cdots\ 0\ 1\ 2\ ③\ 4\\
2 & \cdots\ 5\ 2\ ⑧\ 2\ 3\\
3 & \cdots\ 9\ ④\ 3\ 2\ 1\\
\vdots & \qquad\ \vdots\\
\hline
? & \cdots\ 1\ 5\ 9\ 4\ 2
\end{array}
$$

What is wrong with this argument?

(b) Classify the following sets as either countable or uncountable, with brief justification.

  (i) The set of integers x that solve the equation $3x \equiv 2 \pmod{10}$.

  (ii) The set of real solutions for the equation $x + y = 1$.

  (iii) The set of all irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ (i.e. real numbers that are not rational).