

## Random Variables Intro

Note 15  
Note 19

**Random Variable:** A random variable  $X$  is a function from  $\Omega \rightarrow \mathbb{R}$ , mapping the possible outcomes to real numbers. Note that this function itself is not random; the *outcomes* are random. We define

$$\mathbb{P}[X = k] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = k\}].$$

**Distribution** of a random variable: the set of all  $(k, \mathbb{P}[X = k])$ , describing the probability of attaining each value of the random variable.

**Bernoulli Distribution:**  $X \sim \text{Bernoulli}(p)$ ;  $X$  represents the outcome of a biased coin flip.  $X$  is oftentimes also called an *indicator random variable* of an event with probability  $p$ . The distribution is described by the following:

$$\mathbb{P}[X = k] = \begin{cases} p & \text{if } k = 1 \\ 1 - p & \text{if } k = 0 \end{cases}$$

**Binomial Distribution:**  $X \sim \text{Binomial}(n, p)$ ;  $X$  represents the number of successes in  $n$  independent trials, where  $p$  is the probability of success in each trial.

**Geometric Distribution:**  $X \sim \text{Geometric}(p)$ ;  $X$  represents the number of independent trials until the first success (including the success), where  $p$  is the probability of success in each trial.

**Poisson Distribution:**  $X \sim \text{Poisson}(\lambda)$ ;  $X$  represents the number of occurrences of an event in one unit of time, if on average there are  $\lambda$  occurrences in one unit of time. The distribution is described by the following:

$$\mathbb{P}[X = k] = \frac{\lambda^k}{k!} e^{-\lambda}$$

Further, if  $X \sim \text{Poisson}(\lambda_x)$  and  $Y \sim \text{Poisson}(\lambda_y)$  are independent, then  $X + Y \sim \text{Poisson}(\lambda_x + \lambda_y)$ .

# 1 Head Count

Note 15

Consider a coin with  $\mathbb{P}[\text{Heads}] = 2/5$ . Suppose you flip the coin 20 times, and define  $X$  to be the number of heads.

- (a) What is  $\mathbb{P}[X = k]$ , for some  $0 \leq k \leq 20$ ? Express your answer in terms of  $k$ . (Do not just copy down a formula—re-derive it yourself!)
- (b) What is the name of the distribution of  $X$ , and what are its parameters?
- (c) What is  $\mathbb{P}[X \geq 1]$ ? *Hint: You should be able to do this without a summation.*
- (d) What is  $\mathbb{P}[12 \leq X \leq 14]$ ?
- (e) Now consider a second coin also with  $\mathbb{P}[\text{Heads}] = 2/5$ . Suppose you flip this second coin 30 times, and define  $Y$  to be the number of heads. What is the distribution of the *total* number of heads among these two coins, i.e. what is the distribution of  $X + Y$ ?

## 2 Head Count II

Note 19

Consider a coin with  $\mathbb{P}[\text{Heads}] = 3/4$ . Suppose you flip the coin until you see heads for the first time, and define  $X$  to be the number of times you flipped the coin.

- (a) What is  $\mathbb{P}[X = k]$ , for some  $k \geq 1$ ? Express your answer in terms of  $k$ . (Do not just copy down a formula—re-derive it yourself!)
- (b) What is the name of the distribution of  $X$ , and what are its parameters?
- (c) What is  $\mathbb{P}[X > k]$ , for some  $k \geq 0$ ? (You should not have any summations.)
- (d) What is  $\mathbb{P}[X < k]$ , for some  $k \geq 1$ ? (You should not have any summations.)
- (e) What is  $\mathbb{P}[X > k \mid X > m]$ , for some  $k \geq m \geq 0$ ? Show that your answer is equal to  $\mathbb{P}[X > k - m]$ . Why do we call this the memoryless property?
- (f) Suppose  $X \sim \text{Geometric}(p)$  and  $Y \sim \text{Geometric}(q)$  are independent. Find the distribution of  $\min(X, Y)$  and justify your answer.

### 3 Shuttles and Taxis at Airport

Note 19

In front of terminal 3 at San Francisco Airport is a pickup area where shuttles and taxis arrive according to a Poisson distribution. The shuttles arrive at a rate  $\lambda_1 = 1/20$  (i.e. 1 shuttle per 20 minutes) and the taxis arrive at a rate  $\lambda_2 = 1/10$  (i.e. 1 taxi per 10 minutes) starting at 00:00. The shuttles and the taxis arrive independently.

- (a) What is the distribution of the following:
  - (i) The number of taxis that arrive between times 00:00 and 00:20?
  - (ii) The number of shuttles that arrive between times 00:00 and 00:20?
  - (iii) The total number of pickup vehicles that arrive between times 00:00 and 00:20?
- (b) What is the probability that exactly 1 shuttle and 3 taxis arrive between times 00:00 and 00:20?
- (c) Given that exactly 1 pickup vehicle arrived between times 00:00 and 00:20, what is the conditional probability that this vehicle was a taxi?
- (d) Suppose you reach the pickup area at 00:20. You learn that you missed 3 taxis and 1 shuttle in those 20 minutes. What is the probability that you need to wait for more than 10 mins until either a shuttle or a taxi arrives?