Due: Saturday, 11/16, 4:00 PM
Grace period until Saturday, 11/16, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Coupon Collector Variance

Note 19

It's that time of the year again—Safeway is offering its Monopoly Card promotion. Each time you visit Safeway, you are given one of n different Monopoly Cards with equal probability. You need to collect them all to redeem the grand prize.

Let X be the number of visits you have to make before you can redeem the grand prize. Show that $\text{Var}(X) = n^2\left(\sum_{i=1}^{n} i^{-2}\right) - \mathbb{E}[X]$.

2 Diversify Your Hand

Note 15
Note 16

You are dealt 5 cards from a standard 52 card deck. Let X be the number of distinct values in your hand. For instance, the hand (A, A, A, 2, 3) has 3 distinct values.

(a) Calculate $\mathbb{E}[X]$. (Hint: Consider indicator variables X_i representing whether i appears in the hand.)

(b) Calculate $\text{Var}(X)$. The answer expression will be quite involved; you do not need to simplify anything.

3 Double-Check Your Intuition Again

Note 16

(a) You roll a fair six-sided die and record the result X. You roll the die again and record the result Y.

(i) What is $\text{cov}(X+Y, X-Y)$?

(ii) Prove that $X+Y$ and $X-Y$ are not independent.

For each of the problems below, if you think the answer is "yes" then provide a proof. If you think the answer is "no", then provide a counterexample.

(b) If X is a random variable and $\text{Var}(X) = 0$, then must X be a constant?

(c) If X is a random variable and c is a constant, then is $\text{Var}(cX) = c\,\text{Var}(X)$?

(d) If A and B are random variables with nonzero standard deviations and $\text{Corr}(A,B) = 0$, then are A and B independent?

(e) If X and Y are not necessarily independent random variables, but $\text{Corr}(X,Y) = 0$, and X and Y have nonzero standard deviations, then is $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$?

(f) If X and Y are random variables then is $\mathbb{E}[\max(X,Y)\min(X,Y)] = \mathbb{E}[XY]$?

(g) If X and Y are independent random variables with nonzero standard deviations, then is

$$\text{Corr}(\max(X,Y), \min(X,Y)) = \text{Corr}(X,Y)?$$

4 Dice Games

Note 20

(a) Alice rolls a die until she gets a 1. Let X be the number of total rolls she makes (including the last one), and let Y be the number of rolls on which she gets an even number. Compute $\mathbb{E}[Y \mid X = x]$, and use it to calculate $\mathbb{E}[Y]$.

(b) Bob plays a game in which he starts off with one die. At each time step, he rolls all the dice he has. Then, for each die, if it comes up as an odd number, he puts that die back, and adds a number of dice equal to the number displayed to his collection. (For example, if he rolls a one on the first time step, he puts that die back along with an extra die.) However, if it comes up as an even number, he removes that die from his collection.

Compute the expected number of dice Bob will have after n time steps. (Hint: compute the value of $\mathbb{E}[X_k \mid X_{k-1} = m]$ to derive a recursive expression for X_k, where X_i is the random variable representing the number of dice after i time steps.)

5 Iterated Expectation

Note 20

In this question, we will try to achieve more familiarity with the law of iterated expectation.

(a) You lost your phone charger! It will take D days for the new phone charger you ordered to arrive at your house (here, D is a random variable). Suppose that on day i, the amount of battery you lose is B_i, where $\mathbb{E}[B_i] = \beta$. Let $B = \sum_{i=1}^{D} B_i$ be the total amount of battery drained between now and when your new phone charger arrives. Apply the law of iterated expectation to show that $\mathbb{E}[B] = \beta\,\mathbb{E}[D]$.

(Here, the law of iterated expectation has a very clear interpretation: the amount of battery you expect to drain is the average number of days it takes for your phone charger to arrive, multiplied by the average amount of battery drained per day.)

(b) Consider now the setting of independent Bernoulli trials, each with probability of success p. Let S_i be the number of successes in the first i trials. Compute $\mathbb{E}[S_m \mid S_n = k]$.

(You will need to consider three cases based on whether $m > n$, $m = n$, or $m < n$. Try using your intuition rather than proceeding by calculations.)