Due: Monday 12/2, 4:00 PM
Grace period until Monday 12/2, 11:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Predictable Gaussians

Note 21

Let Y be the result of a fair coin flip, and X be a normally distributed random variable with parameters dependent on Y. That is, if $Y = 1$, then $X \sim N(\mu_1, \sigma_1^2)$, and if $Y = 0$, then $X \sim N(\mu_0, \sigma_0^2)$.

(a) Sketch the two distributions of X overlaid on the same graph for the following cases:

(i) $\sigma_0^2 = \sigma_1^2, \mu_0 \neq \mu_1$

(ii) $\sigma_0^2 \neq \sigma_1^2, \mu_0 = \mu_1$

(b) Bayes' rule for mixed distributions can be formulated as $\mathbb{P}[Y = 1 \mid X = x] = \frac{\mathbb{P}[Y=1] f_{X|Y=1}(x)}{f_X(x)}$ where Y is a discrete distribution and X is a continuous distribution. Compute $\mathbb{P}[Y = 1 \mid X = x]$, and show that this can be expressed in the form of $\frac{1}{1+e^\gamma}$ for some expression γ. (Hint: any value z can be equivalently expressed as $e^{\ln(z)}$)

(c) In the special case where $\sigma_0^2 = \sigma_1^2$ find a simple expression for the value of x where $\mathbb{P}[Y = 1 \mid X = x] = \mathbb{P}[Y = 0 \mid X = x] = 1/2$, and interpret what the expression represents. (Hint: the identity $(a+b)(a-b) = a^2 - b^2$ may be useful)

2 Moments of the Gaussian

Note 21

For a random variable X, the quantity $\mathbb{E}[X^k]$ for $k \in \mathbb{N}$ is called the *kth moment* of the distribution. In this problem, we will calculate the moments of a standard normal distribution.

(a) Prove the identity

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{tx^2}{2}\right) \mathrm{d}x = t^{-1/2}$$

for $t > 0$.

Hint: Consider a normal distribution with variance $\frac{1}{t}$ and mean 0.

(b) For the rest of the problem, X is a standard normal distribution (with mean 0 and variance 1). Use part (a) to compute $\mathbb{E}[X^{2k}]$ for $k \in \mathbb{N}$.

Hint: Try differentiating both sides with respect to t, k times. You may use the fact that we can differentiate under the integral without proof.

(c) Compute $\mathbb{E}[X^{2k+1}]$ for $k \in \mathbb{N}$.

3 Chebyshev's Inequality vs. Central Limit Theorem

Note 17
Note 21

Let n be a positive integer. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with the following distribution:

$$\mathbb{P}[X_i = -1] = \frac{1}{12}; \qquad \mathbb{P}[X_i = 1] = \frac{9}{12}; \qquad \mathbb{P}[X_i = 2] = \frac{2}{12}.$$

(a) Calculate the expectations and variances of X_1, $\sum_{i=1}^n X_i$, $\sum_{i=1}^n (X_i - \mathbb{E}[X_i])$, and

$$Z_n = \frac{\sum_{i=1}^n (X_i - \mathbb{E}[X_i])}{\sqrt{n/2}}.$$

(b) Use Chebyshev's Inequality to find an upper bound b for $\mathbb{P}[|Z_n| \geq 2]$.

(c) Use b from the previous part to bound $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$.

(d) As $n \to \infty$, what is the distribution of Z_n?

(e) We know that if $Z \sim \mathcal{N}(0,1)$, then $\mathbb{P}[|Z| \leq 2] = \Phi(2) - \Phi(-2) \approx 0.9545$. As $n \to \infty$, provide approximations for $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$.

4 (Optional) LLSE and Graphs

Note 20

Consider a graph with n vertices numbered 1 through n, where n is a positive integer ≥ 2. For each pair of distinct vertices, we add an undirected edge between them independently with probability p. Let D_1 be the random variable representing the degree of vertex 1, and let D_2 be the random variable representing the degree of vertex 2.

(a) Compute $\mathbb{E}[D_1]$ and $\mathbb{E}[D_2]$.

(b) Compute $\mathrm{Var}(D_1)$.

(c) Compute $\mathrm{cov}(D_1, D_2)$.

(d) Using the information from the first three parts, what is $L(D_2 \mid D_1)$?