CS 70 Fall 2024 — Discrete Mathematics and Probability Theory — Hug, Rao — **HW 14**

Due: Monday 12/9, 4:00 PM
Grace period until Monday 12/9, 11:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Balls in Bins Estimation

Note 20

We throw $n > 0$ balls into $m \geq 2$ bins. Let X and Y represent the number of balls that land in bin 1 and 2 respectively.

(a) Calculate $\mathbb{E}[Y \mid X]$. [*Hint*: Your intuition may be more useful than formal calculations.]

(b) What is $L[Y \mid X]$ (where $L[Y \mid X]$ is the best linear estimator of Y given X)? [*Hint*: Your justification should be no more than two or three sentences, no calculations necessary! Think carefully about the meaning of the conditional expectation.]

(c) Unfortunately, your friend is not convinced by your answer to the previous part. Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(d) Compute $\mathrm{Var}(X)$.

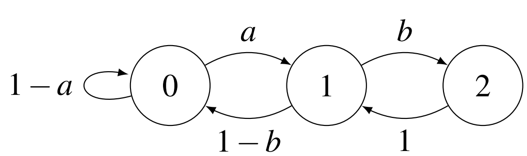
(e) Compute $\mathrm{cov}(X,Y)$.

(f) Compute $L[Y \mid X]$ using the formula. Ensure that your answer is the same as your answer to part (b).

2 Analyze a Markov Chain

Note 22

Consider a Markov chain with the state diagram shown below where $a, b \in (0,1)$.



Here, we let $X(n)$ denote the state at time n.

(a) Is this Markov chain irreducible? Is this Markov chain aperiodic? Justify your answers.

(b) Calculate $\mathbb{P}[X(1) = 1, X(2) = 0, X(3) = 0, X(4) = 1 \mid X(0) = 0]$.

(c) Calculate the invariant distribution. Do all initial distributions converge to this invariant distribution? Justify your answer.

3 A Bit of Everything

Note 22 Suppose that $X_0, X_1, \ldots$ is a Markov chain with finite state space $S = \{1, 2, \ldots, n\}$, where $n > 2$, and transition matrix P. Suppose further that

$$\begin{aligned} P(1,i) &= \frac{1}{n} \quad \text{for all states } i \text{ and} \\ P(j, j-1) &= 1 \quad \text{for all states } j \neq 1, \end{aligned}$$

with $P(i, j) = 0$ everywhere else.

(a) Prove that this Markov chain is irreducible and aperiodic.

(b) Suppose you start at state 1. What is the distribution of T, where T is the number of transitions until you leave state 1 for the first time?

(c) Again starting from state 1, what is the expected number of transitions until you reach state n for the first time?

(d) Again starting from state 1, what is the probability you reach state n before you reach state 2?

(e) Compute the stationary distribution of this Markov chain.

4 Playing Blackjack

Note 22 Suppose you start with \$1, and at each turn, you win \$1 with probability p, or lose \$1 with probability $1 - p$. You will continually play games of Blackjack until you either lose all your money, or you have a total of n dollars.

(a) Formulate this problem as a Markov chain.

(b) Let $\alpha(i)$ denote the probability that you end the game with n dollars, given that you started with i dollars.

Notice that for $0 < i < n$, we can write $\alpha(i+1) - \alpha(i) = k(\alpha(i) - \alpha(i-1))$. Find k.

(c) Using part (b), find $\alpha(i)$, where $0 \leq i \leq n$. (You will need to split into two cases: $p = \frac{1}{2}$ or $p \neq \frac{1}{2}$.)

Hint: Try to apply part (b) iteratively, and look at a telescoping sum to write $\alpha(i)$ in terms of $\alpha(1)$. The formula for the sum of a finite geometric series may be helpful when looking at the case where $p \neq \frac{1}{2}$:

$$\sum_{k=0}^{m} a^k = \frac{1-a^{m+1}}{1-a}.$$

Lastly, it may help to use the value of $\alpha(n)$ to find $\alpha(1)$ for the last few steps of the calculation.

(d) As $n \to \infty$, what happens to the probability of ending the game with n dollars, given that you start with i dollars, with the following values of p?

(i) $p > \frac{1}{2}$

(ii) $p = \frac{1}{2}$

(iii) $p < \frac{1}{2}$