Today.

Finish up counting.

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Countabiity.

Some Practice.

How many orderings of letters of CAT?

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3 ways to choose first letter, 2 ways for second, 1 for last.

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$\Longrightarrow 3 \times 2 \times 1$

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$\implies 3 \times 2 \times 1 = 3!$ orderings

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Some Practice.

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Ordered,

Some Practice.

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3 ways to choose first letter, 2 ways for second, 1 for last.

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How many orderings of the letters in ANAGRAM?

Ordered, except for A!

Some Practice.

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How many orderings of the letters in ANAGRAM?

Ordered, except for A!

total orderings of 7 letters.

Some Practice.

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Ordered, except for A!

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Some Practice.

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3 ways to choose first letter, 2 ways for second, 1 for last.

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How many orderings of the letters in ANAGRAM?

Ordered, except for A!

total orderings of 7 letters. 7!
total “extra counts” or orderings of three A’s?

Some Practice.

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3 ways to choose first letter, 2 ways for second, 1 for last.

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How many orderings of the letters in ANAGRAM?

Ordered, except for A!

total orderings of 7 letters. 7!
total "extra counts" or orderings of three A's? 3!

Some Practice.

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Some Practice.

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4 S’s, 4 I’s, 2 P’s.

Some Practice.

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4 S’s, 4 I’s, 2 P’s.
11 letters total.

Some Practice.

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How many orderings of MISSISSIPPI?

4 S's, 4 I's, 2 P's.
11 letters total.
11! ordered objects.

Some Practice.

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11 letters total.
11! ordered objects.
$4! \times 4! \times 2!$ ordered objects per "unordered object"

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11 letters total.
$11!$ ordered objects.
$4! \times 4! \times 2!$ ordered objects per "unordered object"

$\implies \frac{11!}{4!4!2!}$.

Summary.

First rule: $n_1 \times n_2 \cdots \times n_3$.

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When order doesn't matter (sometimes) can divide...

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Sample k times n with replacement and order doesn't matter:
$\binom{k+n-1}{n-1}$.

Sampling...

Sample k items out of n

Sampling...

Sample k items out of n

Without replacement:

Sampling...

Sample k items out of n

Without replacement:

Order matters:

Sampling...

Sample k items out of n

Without replacement:

Order matters: $n \times$

Sampling...

Sample k items out of n

Without replacement:

Order matters: $n \times n-1 \times n-2 \ldots$

Sampling...

Sample k items out of n

Without replacement:

Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1$

Sampling...

Sample k items out of n

Without replacement:

Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$

Order does not matter:

Sampling...

Sample k items out of n

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Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$

Order does not matter:

Second Rule: divide by number of orders

Sampling...

Sample k items out of n

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Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$

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"n choose k"

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Order matters: n

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Order does not matter:

Second Rule: divide by number of orders – "k!"

$\implies \frac{n!}{(n-k)!k!}$.

"n choose k"

With Replacement.

Order matters: $n \times n$

Sampling...

Sample k items out of n

Without replacement:

Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$

Order does not matter:

Second Rule: divide by number of orders – "k!"

$\implies \frac{n!}{(n-k)!k!}$.

"n choose k"

With Replacement.

Order matters: $n \times n \times \ldots n$

Sampling...

Sample k items out of n

Without replacement:

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Order does not matter:

Second Rule: divide by number of orders – "k!"

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"n choose k"

With Replacement.

Order matters: $n \times n \times \ldots n = n^k$

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Problem: depends on how many of each item we chose!

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Problem: depends on how many of each item we chose!

Different number of unordered elts map to each unordered elt.

Sampling...

Sample k items out of n

Without replacement:

Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$

Order does not matter:

Second Rule: divide by number of orders – "k!"

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Unordered elt: $1,2,3$

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Sample k items out of n

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Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$

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Different number of unordered elts map to each unordered elt.

Unordered elt: $1, 2, 3$ 3! ordered elts map to it.

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Sample k items out of n

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Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$

Order does not matter:

Second Rule: divide by number of orders – "k!"

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Order matters: $n \times n \times \ldots n = n^k$

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Different number of unordered elts map to each unordered elt.

Unordered elt: $1,2,3$ $\quad$ 3! ordered elts map to it.

Unordered elt: $1,2,2$

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Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$

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Different number of unordered elts map to each unordered elt.

Unordered elt: $1,2,3$ $3!$ ordered elts map to it.

Unordered elt: $1,2,2$ $\frac{3!}{2!}$ ordered elts map to it.

Sampling...

Sample k items out of n

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Unordered elt: $1,2,3$ $3!$ ordered elts map to it.

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How do we deal with this

Sampling...

Sample k items out of n

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Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$

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Different number of unordered elts map to each unordered elt.

Unordered elt: $1, 2, 3$ 3! ordered elts map to it.
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How do we deal with this mess??

Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.

Splitting 5 dollars..

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Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).

Splitting 5 dollars..

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Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).

Separate Alice's dollars from Bob's and then Bob's from Eve's.

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Five dollars are five stars: $\star\star\star\star\star$.

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Alice: 2, Bob: 1, Eve: 2.

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Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: $\star\star|\star|\star\star$.

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Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: $\star\star|\star|\star\star$.

Alice: 0, Bob: 1, Eve: 4.

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Each split "is" a sequence of stars and bars.

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Each split "is" a sequence of stars and bars.
Each sequence of stars and bars "is" a split.

Counting Rule: if there is a one-to-one mapping between two sets they have the same size!

Stars and Bars.

How many different 5 star and 2 bar diagrams?

Stars and Bars.

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$| \star | \star \star \star \star$.

Stars and Bars.

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$| \star | \star \star \star \star$.

7 positions in which to place the 2 bars.

$\_ \; \_ \; \_ \; \_ \; \_ \; \_ \; \_$

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Alice: 0; Bob 1; Eve: 4

Stars and Bars.

How many different 5 star and 2 bar diagrams?

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$\binom{7}{2}$ ways to split 5 dollars among 3 people.

Stars and Bars.

Ways to add up n numbers to sum to k?

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Or: k unordered choices from set of n possibilities with replacement.
Sample with replacement where order doesn't matter.

Counting basics.

First rule: $n_1 \times n_2 \cdots \times n_3$.

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Sample with replacement and order doesn't matter: $\binom{k+n-1}{n-1}$.

Bijection: sums to 'k' $\rightarrow$ stars and bars.

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$|S| = |T| = \binom{7}{2}$.

Stars and Bars Poll

Mark whats correct.

(A) ways to split n dollars among k: $\binom{n+k-1}{k-1}$

(B) ways to split k dollars among n: $\binom{k+n-1}{n-1}$

(C) ways to split 5 dollars among 3: $\binom{7}{5}$

(D) ways to split 5 dollars among 3: $\binom{5+3-1}{3-1}$

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All correct.

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Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?

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$\square$

Pascal’s Triangle

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$$\begin{array}{ccc} & 0 & \\ 1 & & 1 \end{array}$$

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$$\begin{array}{ccccc} & & 0 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

Pascal's Triangle

			0			
		1		1		
	1		2		1	
1		3		3		1

Pascal's Triangle

				0				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Pascal's Triangle

0

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Row n: coefficients of $(a+b)^n = (a+b)(a+b)\cdots(a+b)$.

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Foil (4 terms)

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$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
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				0				
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2^n terms: choose 1 or x from each term $(1+x)$.

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$$\begin{array}{ccccccccc} &&&& 0 &&&& \\ &&& 1 && 1 &&& \\ && 1 && 2 && 1 && \\ & 1 && 3 && 3 && 1 & \\ 1 && 4 && 6 && 4 && 1 \end{array}$$

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Foil (4 terms) on steroids:

$(a+b)^2 = a^2 + ab + ba + b^2$

$aaa + aab + aba + abb + baa + bab + bba + bbb$.

2^n terms: choose 1 or x from each term $(1+x)$.

Pascal's Triangle

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Pascal's rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$?

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Theorem: ${n+1 \choose k} = {n \choose k} + {n \choose k-1}$.

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How many contain the first element?

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Chose first element, need $k-1$ more from remaining n elements.

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So, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. $\square$

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Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S **and** T**,** $|S \cup T| = |S| + |T|$
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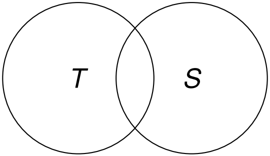

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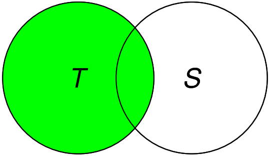

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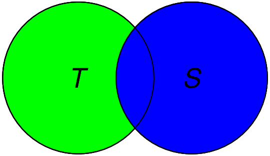

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In S. $\implies + \ |S|$

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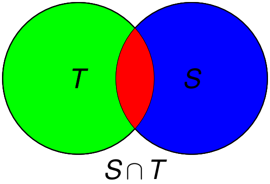

In T. $\implies |T|$
In S. $\implies + \ |S|$
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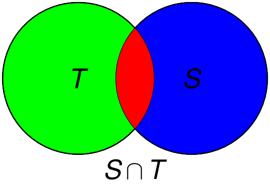

In T. $\implies$ $|T|$
In S. $\implies$ $+ \; |S|$
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Subtract. $\implies$ $-|S \cap T|$

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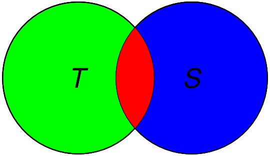

$S \cap T$

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Inclusion/Exclusion Rule: For any S and T,
$|S \cup T| = |S| + |T| - |S \cap T|$.

Example: How many 10-digit phone numbers have 7 as their first or second digit?

S = phone numbers with 7 as first digit. $|S| = 10^9$

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
Used to reason about all subsets
by adding number of subsets of size 1, 2, 3,...

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T = phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T$ = phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

Inclusion/Exclusion

$$|A_1 \cup \cdots \cup A_n| =$$
$$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|.$$

Inclusion/Exclusion

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Idea: For $n = 3$ how many times is an element counted?

Inclusion/Exclusion

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x counted once for $|A_i|$ and once for $|A_j|$.

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Idea: For $n = 3$ how many times is an element counted?
Consider $x \in A_i \cap A_j$.
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Total: 2 -1 = 1.

Inclusion/Exclusion

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Total: 2 -1 = 1.

Consider $x \in A_1 \cap A_2 \cap A_3$

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
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Total: 2 -1 = 1.

Consider $x \in A_1 \cap A_2 \cap A_3$

x counted once in each term: $|A_1|, |A_2|, |A_3|$.

Inclusion/Exclusion

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Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

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Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|.$

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x added once in $|A_1 \cap A_2 \cap A_3|$.

Total: 3 - 3 + 1 = 1.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

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Total: 3 - 3 + 1 = 1.

Formulaically: x is in intersection of three sets.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

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Total: 2 -1 = 1.

Consider $x \in A_1 \cap A_2 \cap A_3$

x counted once in each term: $|A_1|, |A_2|, |A_3|$.

x subtracted once in terms: $|A_1 \cap A_3|$, $|A_1 \cap A_2|$, $|A_2 \cap A_3|$.

x added once in $|A_1 \cap A_2 \cap A_3|$.

Total: 3 - 3 + 1 = 1.

Formulaically: x is in intersection of three sets.

$\binom{3}{1}$ for terms of form $|A_i|$, $\binom{3}{2}$ for terms of form $|A_i \cap A_j|$.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

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Total: 2 -1 = 1.

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Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
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x counted once in each term: $|A_1|, |A_2|, |A_3|$.

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$\binom{3}{3}$ for terms of form $|A_i \cap A_j \cap A_k|$.

Total: $\binom{3}{1} - \binom{3}{2} + \binom{3}{3} = 1$.

Inclusion/Exclusion

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Idea: how many times is each element counted?

Element x in m sets: $x \in A_{i_1} \cap A_{i_2} \cdots \cap A_{i_m}$.

Inclusion/Exclusion

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Counted $\binom{m}{i}$ times in ith summation.

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Binomial Theorem:
$(x+y)^m = \binom{m}{0}x^m + \binom{m}{1}x^{m-1}y + \binom{m}{2}x^{m-2}y^2 + \cdots \binom{m}{m}y^m$.

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Proof: m factors in product: $(x+y)(x+y)\cdots(x+y)$.

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Get a term $x^{m-i} y^i$ by choosing i factors to use for y.

Inclusion/Exclusion

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Inclusion/Exclusion

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$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

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Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
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For $x = 1, y = -1$,

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

Idea: how many times is each element counted?
Element x in m sets: $x \in A_{i_1} \cap A_{i_2} \cdots \cap A_{i_m}$.
Counted $\binom{m}{i}$ times in ith summation.
Total: $\binom{m}{1} - \binom{m}{2} + \binom{m}{3} \cdots + (-1)^{m-1}\binom{m}{m}$.

Binomial Theorem:
$(x+y)^m = \binom{m}{0}x^m + \binom{m}{1}x^{m-1}y + \binom{m}{2}x^{m-2}y^2 + \cdots \binom{m}{m}y^m$.
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Get a term $x^{m-i}y^i$ by choosing i factors to use for y.
are $\binom{m}{i}$ ways to choose factors where y is provided. □

For $x = 1, y = -1$,
$0 = (1-1)^m = \binom{m}{0} - \binom{m}{1} + \binom{m}{2} \cdots + (-1)^m \binom{m}{m}$

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|.$

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Element x in m sets: $x \in A_{i_1} \cap A_{i_2} \cdots \cap A_{i_m}$.

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Total: $\binom{m}{1} - \binom{m}{2} + \binom{m}{3} \cdots + (-1)^{m-1} \binom{m}{m}$.

Binomial Theorem:

$(x+y)^m = \binom{m}{0} x^m + \binom{m}{1} x^{m-1} y + \binom{m}{2} x^{m-2} y^2 + \cdots \binom{m}{m} y^m.$

Proof: m factors in product: $(x+y)(x+y)\cdots(x+y)$.

Get a term $x^{m-i} y^i$ by choosing i factors to use for y.

are $\binom{m}{i}$ ways to choose factors where y is provided. $\square$

For $x = 1, y = -1$,

$0 = (1-1)^m = \binom{m}{0} - \binom{m}{1} + \binom{m}{2} \cdots + (-1)^m \binom{m}{m}$

$\implies \quad 1 = \binom{m}{0} = \binom{m}{1} - \binom{m}{2} \cdots + (-1)^{m-1} \binom{m}{m}.$

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

Idea: how many times is each element counted?

Element x in m sets: $x \in A_{i_1} \cap A_{i_2} \cdots \cap A_{i_m}$.

Counted $\binom{m}{i}$ times in ith summation.

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Each element counted once!

Summary.

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Count with order:

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Disjoint – so add!

Poll: How big is infinity?

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Mark what's true.

(A) There are more real numbers than natural numbers.
(B) There are more rational numbers than natural numbers.
(C) There are more integers than natural numbers.
(D) pairs of natural numbers $>>$ natural numbers.

Same Size. Poll.

Two sets are the same size?

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(A) Bijection between the sets.
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How to count?

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0,

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0, 1,

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How to count?

0, 1, 2,

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How to count?

0, 1, 2, 3,

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0, 1, 2, 3, ...

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S is **countable** if there is a bijection between S and some subset of $\mathbb{N}$.

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Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

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All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.

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Never get to 1.

More fractions?

Enumerate the rational numbers in order...

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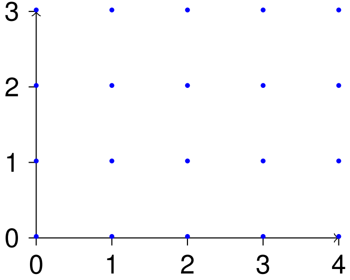

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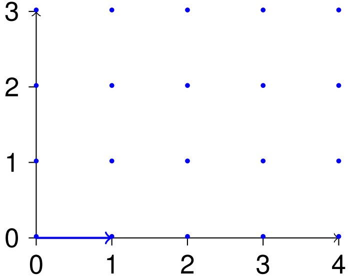

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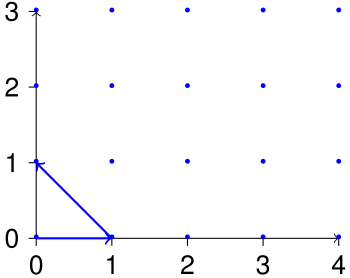
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



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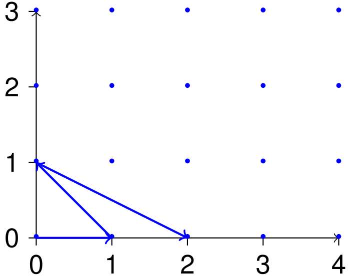

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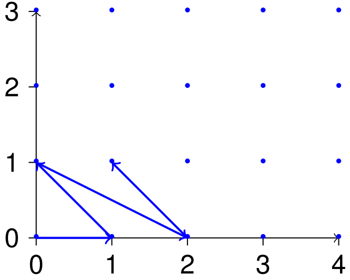

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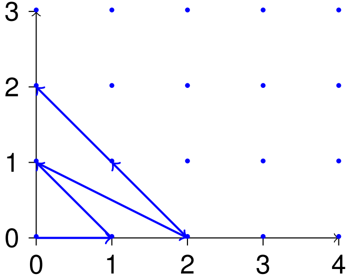

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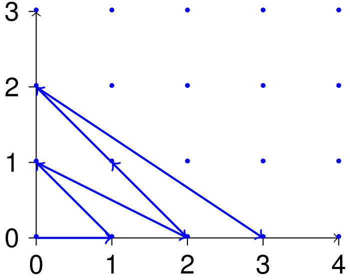

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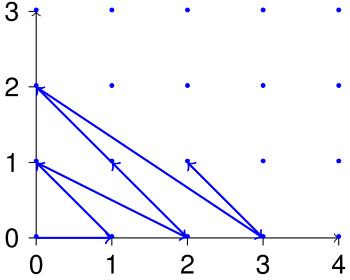

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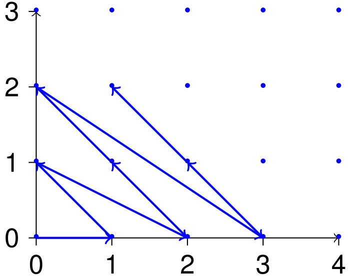

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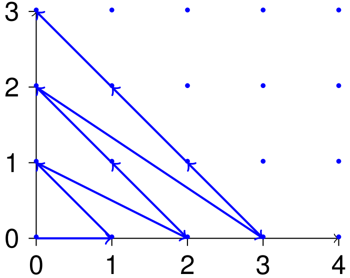

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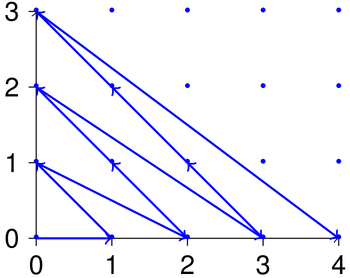

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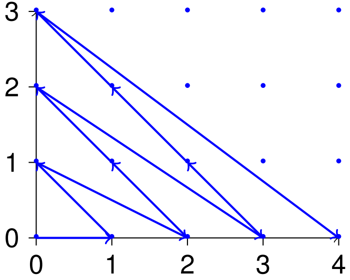

The pair (a,b), is in first $\approx (a+b+1)(a+b)/2$ elements of list!

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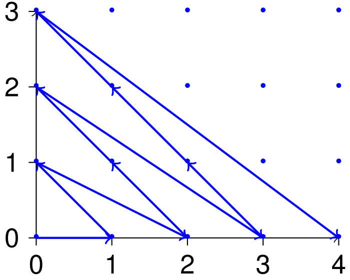

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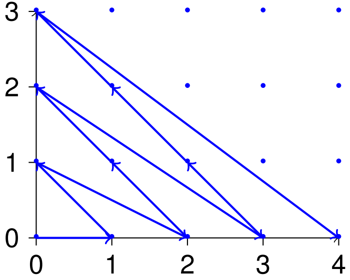

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Countably infinite.

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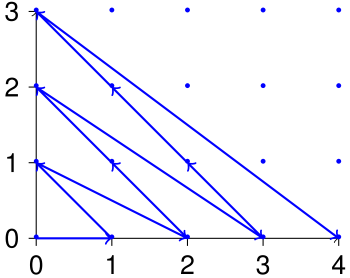

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Same size as the natural numbers!!

Poll.

Enumeration to get bijection with naturals?

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(A) Integers: First all negatives, then positives.
(B) Integers: By absolute value, break ties however.
(C) Pairs of naturals: by sum of values, break ties however.
(D) Pairs of naturals: by value of first element.
(E) Pairs of integers: by sum of values, break ties.
(F) Pairs of integers: by sum of absolute values, break ties.

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Enumeration to get bijection with naturals?

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(B),(C), (F).

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Real numbers..

Real numbers are same size as integers?

The reals.

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Are the set of reals countable?

Lets consider the reals $[0, 1]$.

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Diagonalization.

If countable, there a listing, L contains all reals.

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⋮

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Construct “diagonal” number: .77

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Subset $[0,1]$ is not countable!!

All reals?

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All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

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If reals are countable then so is $[0, 1]$.

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1. Assume that a set S can be enumerated.

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6. Contradiction.

Another diagonalization.

The set of all subsets of N.

Another diagonalization.

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Example subsets of N: $\{0\}$,

Another diagonalization.

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There is a listing, L, that contains all subsets of N.

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$\implies$ D is not in the listing.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.
otherwise $i \notin D$.

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(The set of all subsets of S, is the **powerset** of N.)

Poll: diagonalization Proof.

Mark parts of proof.

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(A) Integers are larger than naturals cuz obviously.
(B) Integers are countable cuz, interleaving bijection.
(C) Reals are uncountable cuz obviously!
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(B), (C)?, (D), (E)

The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

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There is no set with cardinality between the naturals and the reals.

First of Hilbert's problems!

Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

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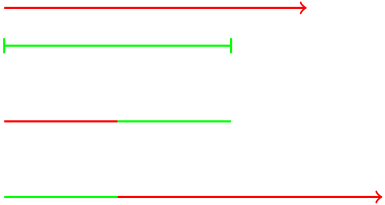

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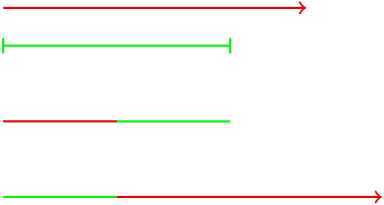

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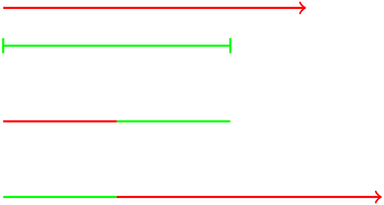

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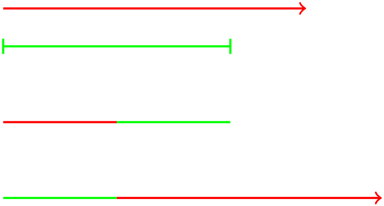

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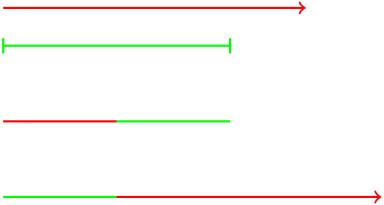

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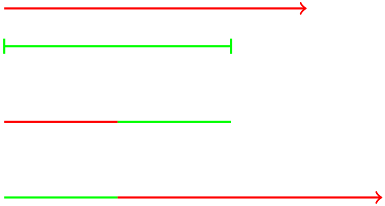

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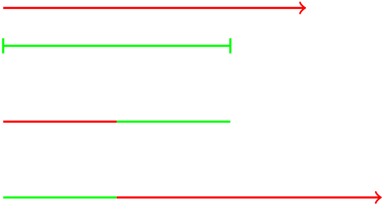

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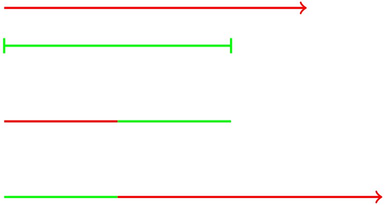

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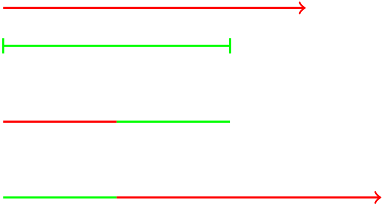

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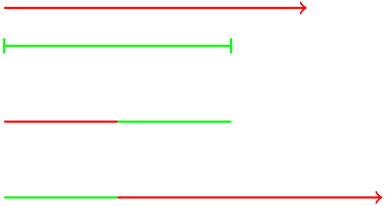

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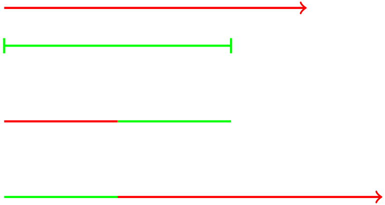

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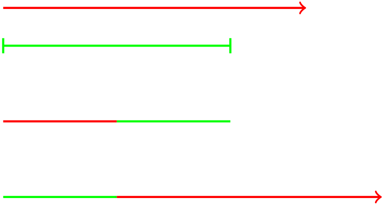

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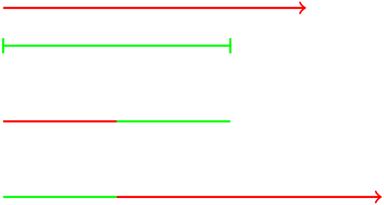

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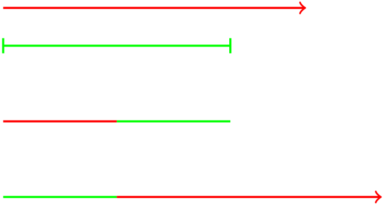

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$[0,1]$ is same cardinality as nonnegative reals!

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Are real numbers even real?

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Recall: powerset of the naturals is not countable.

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Uh oh....