Lecture 7. Outline.

1. Modular Arithmetic.

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   Clock Math!!!

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3. Euclid's GCD Algorithm.

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   A little tricky here!

Hypercubes.

Complete graphs, really connected!

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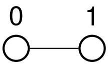

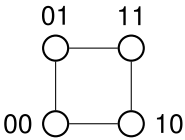

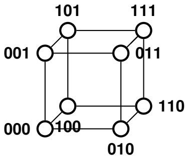

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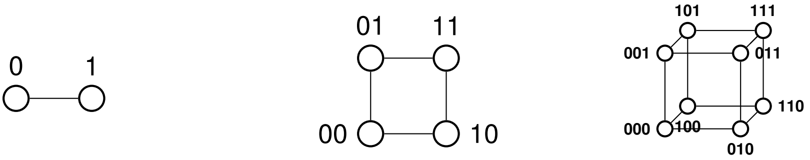

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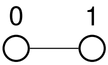

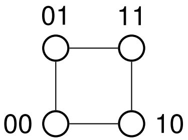

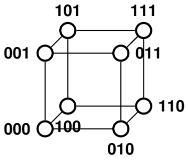

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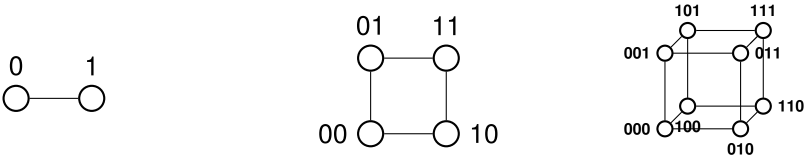

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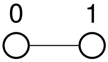

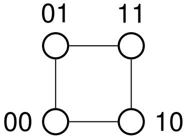

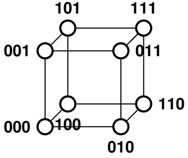

2^n vertices. number of n-bit strings!
$n2^{n-1}$ edges.
2^n vertices each of degree n

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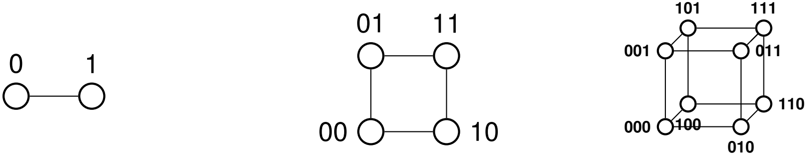

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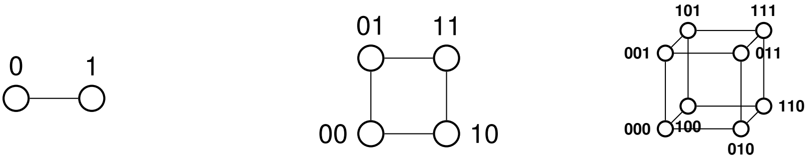

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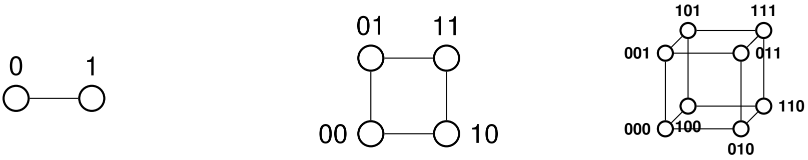

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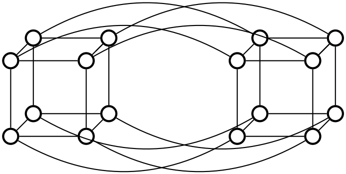

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$(S, V - S)$ is cut.

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Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

Proof of Large Cuts.

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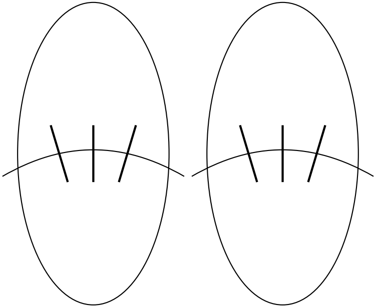

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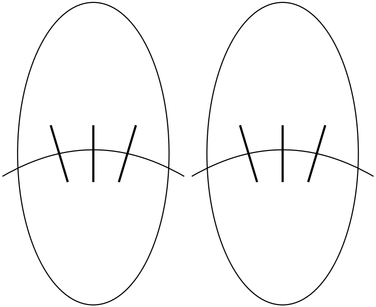

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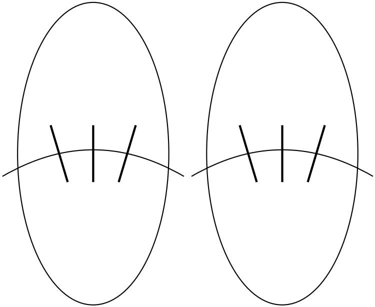

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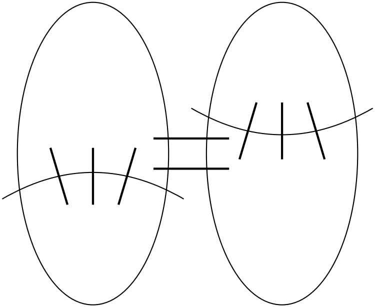

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$S = S_0 \cup S_1$ where S_0 in first, and S_1 in other.

Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$
Both S_0 and S_1 are small sides. So by induction.
Edges cut in $H_0 \geq |S_0|$.
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Induction Step

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step.
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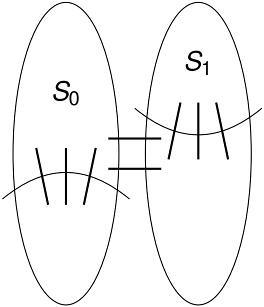

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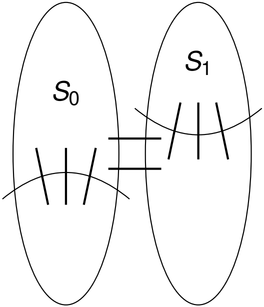

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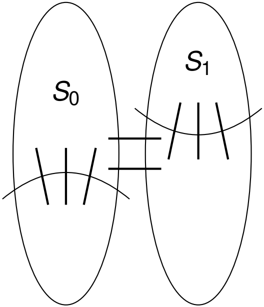

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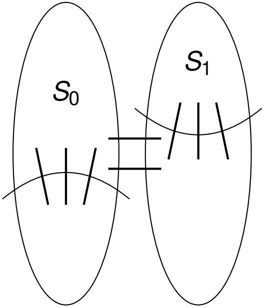

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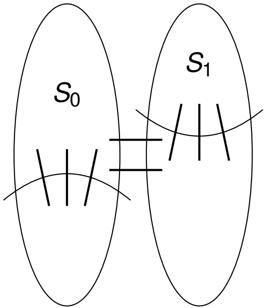

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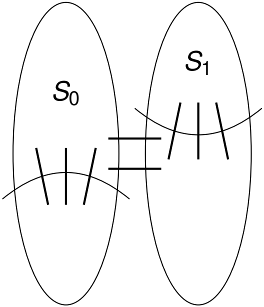

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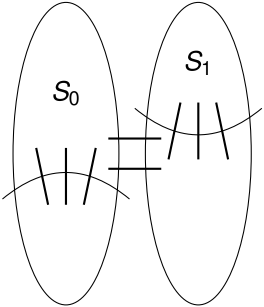

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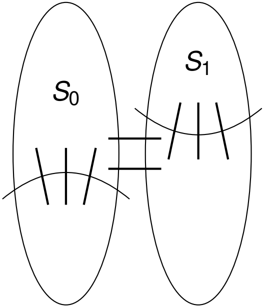

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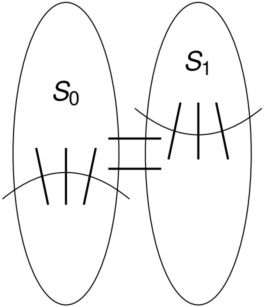

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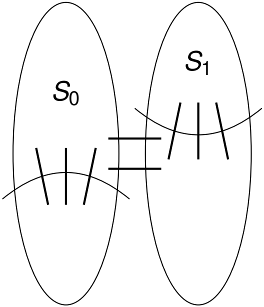

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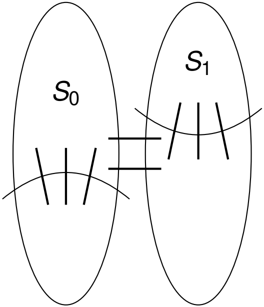

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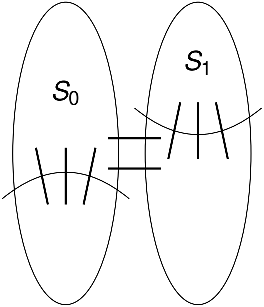

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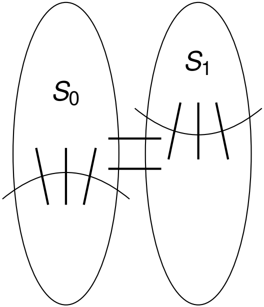

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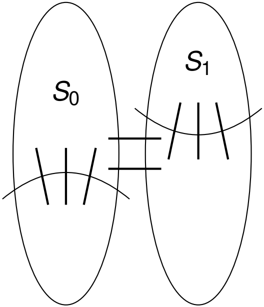

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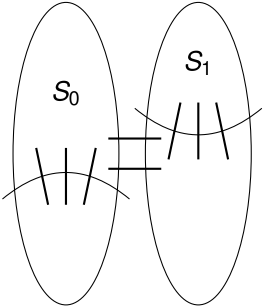

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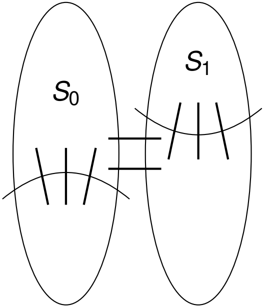

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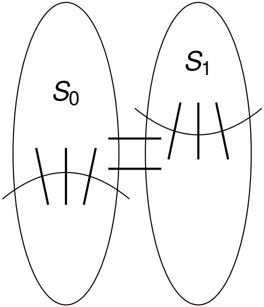

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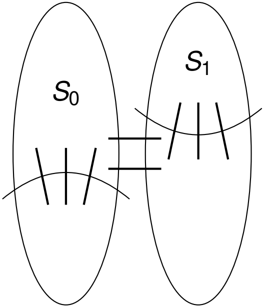

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Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

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Central object of study.

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Graphs:
  Trees: sparsest connected.
  Complete:densest
  Hypercube: middle.

Modular Arithmetic.

Applications: cryptography, error correction.

Key ideas for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y-x)$.

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$\square$

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$\square$

Theorem: Every number $n \geq 2$ can be represented as a product of primes.

Proof: Either prime, or $n = a \times b$, and use strong induction. $\square$
(Uniqueness? Later.)

Poll

What did we use in our proofs of key ideas?

(A) Distributive Property of multiplication over addition.
(B) Euler's formula.
(C) The definition of a prime number.
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(A) and (C)

Next Up.

Modular Arithmetic.

Clock Math

If it is 1:00 now.

Clock Math

If it is 1:00 now.
What time is it in 2 hours?

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours?

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours?

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Clock Math

If it is 1:00 now.

- What time is it in 2 hours? 3:00!
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  - Actually 4:00.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock Math

If it is 1:00 now.
- What time is it in 2 hours? 3:00!
- What time is it in 5 hours? 6:00!
- What time is it in 15 hours? 16:00!
  - Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
- Clock time equivalent up to to addition/subtraction of 12.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours?

Clock Math

If it is 1:00 now.

- What time is it in 2 hours? 3:00!
- What time is it in 5 hours? 6:00!
- What time is it in 15 hours? 16:00!
  - Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00!

Clock Math

If it is 1:00 now.

- What time is it in 2 hours? 3:00!
- What time is it in 5 hours? 6:00!
- What time is it in 15 hours? 16:00!
  - Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

$101 = 12 \times 8 + 5$.

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.

Clock Math

If it is 1:00 now.

- What time is it in 2 hours? 3:00!
- What time is it in 5 hours? 6:00!
- What time is it in 15 hours? 16:00!
  - Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

$101 = 12 \times 8 + 5$.

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Clock time equivalent up to addition of any integer multiple of 12.

Clock Math

If it is 1:00 now.

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$101 = 12 \times 8 + 5$.

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Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in $\{12, 1, \ldots, 11\}$

Clock Math

If it is 1:00 now.

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$101 = 12 \times 8 + 5$.

5 is the same as 101 for a 12 hour clock system.

Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in $\{12, 1, \ldots, 11\}$

(Almost remainder, except for 12 and 0 are equivalent.)

Day of the week.

This is Thursday is September 18, 2024.

Day of the week.

This is Thursday is September 18, 2024.
What day is it a year from now?

Day of the week.

This is Thursday is September 18, 2024.
What day is it a year from now? on September 18, 2025?

Day of the week.

This is Thursday is September 18, 2024.

What day is it a year from now? on September 18, 2025?

Number days.

Day of the week.

This is Thursday is September 18, 2024.

What day is it a year from now? on September 18, 2025?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Day of the week.

This is Thursday is September 18, 2024.

What day is it a year from now? on September 18, 2025?

Number days.

0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Day of the week.

This is Thursday is September 18, 2024.

What day is it a year from now? on September 18, 2025?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

Day of the week.

This is Thursday is September 18, 2024.
  What day is it a year from now? on September 18, 2025?
    Number days.
      0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.
  5 days from then.

Day of the week.

This is Thursday is September 18, 2024.
  What day is it a year from now? on September 18, 2025?
    Number days.
      0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
  5 days from then. day 9

Day of the week.

This is Thursday is September 18, 2024.

What day is it a year from now? on September 18, 2025?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2

Day of the week.

This is Thursday is September 18, 2024.

What day is it a year from now? on September 18, 2025?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

Day of the week.

This is Thursday is September 18, 2024.

What day is it a year from now? on September 18, 2025?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then.

Day of the week.

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What day is it a year from now? on September 18, 2025?

Number days.

0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29

Day of the week.

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5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1.

Day of the week.

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Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

Day of the week.

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Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
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two days are equivalent up to addition/subtraction of multiple of 7.

Day of the week.

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11 days from then

Day of the week.

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5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1

Day of the week.

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  What day is it a year from now? on September 18, 2025?
    Number days.
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    11 days from then is day 1 which is Monday!

Day of the week.

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What day is it a year from then?

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two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year.

Day of the week.

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11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day of the week.

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25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+366 or day 370.

Day of the week.

This is Thursday is September 18, 2024.
  What day is it a year from now? on September 18, 2025?
    Number days.
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Today: day 4.
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Next year is not a leap year. So 365 days from then.
Day 4+366 or day 370. Leap year.
Smallest representation:

Day of the week.

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11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+366 or day 370. Leap year.

Smallest representation:

subtract 7 until smaller than 7.

Day of the week.

This is Thursday is September 18, 2024.
- What day is it a year from now? on September 18, 2025?
  - Number days.
    - 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

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- 5 days from then. day 9 or day 2 or Tuesday.
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  - two days are equivalent up to addition/subtraction of multiple of 7.
  - 11 days from then is day 1 which is Monday!

What day is it a year from then?
- Next year is not a leap year. So 365 days from then.
- Day 4+366 or day 370. Leap year.

Smallest representation:
- subtract 7 until smaller than 7.
- divide and get remainder.

Day of the week.

This is Thursday is September 18, 2024.
What day is it a year from now? on September 18, 2025?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+366 or day 370. Leap year.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
370/7

Day of the week.

This is Thursday is September 18, 2024.
  What day is it a year from now? on September 18, 2025?
    Number days.
      0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.
  5 days from then. day 9 or day 2 or Tuesday.
  25 days from then. day 29 or day 1. $29 = (7)4 + 1$
    two days are equivalent up to addition/subtraction of multiple of 7.
    11 days from then is day 1 which is Monday!

What day is it a year from then?
  Next year is not a leap year. So 365 days from then.
  Day 4+366 or day 370. Leap year.
Smallest representation:
  subtract 7 until smaller than 7.
  divide and get remainder.
  370/7 leaves quotient of 52 and remainder 6.

Day of the week.

This is Thursday is September 18, 2024.

What day is it a year from now? on September 18, 2025?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+366 or day 370. Leap year.

Smallest representation:

subtract 7 until smaller than 7.

divide and get remainder.

370/7 leaves quotient of 52 and remainder 6. $369 = 7(52) + 6$

Day of the week.

This is Thursday is September 18, 2024.

- What day is it a year from now? on September 18, 2025?
  - Number days.
    - 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

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- 25 days from then. day 29 or day 1. $29 = (7)4 + 1$
  - two days are equivalent up to addition/subtraction of multiple of 7.
  - 11 days from then is day 1 which is Monday!

What day is it a year from then?

- Next year is not a leap year. So 365 days from then.
- Day 4+366 or day 370. Leap year.

Smallest representation:

- subtract 7 until smaller than 7.
- divide and get remainder.
- 370/7 leaves quotient of 52 and remainder 6. $369 = 7(52) + 6$
  - or September 18, 2025 is a Saturday.

Day of the week.

This is Thursday is September 18, 2024.

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subtract 7 until smaller than 7.

divide and get remainder.

370/7 leaves quotient of 52 and remainder 6. $369 = 7(52) + 6$

or September 18, 2025 is a Saturday.

Years and years...

80 years?

Years and years...

80 years? 20 leap years.

Years and years...

80 years? 20 leap years. 366×20 days

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7?

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7?

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60$

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Today is day 4.

Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$

Remainder when dividing by 7?

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7$

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or September 18, 2102 is Saturday!

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or September 18, 2102 is Saturday!

Further Simplify Calculation:

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Today is day 4.

Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$

Remainder when dividing by 7? $104 = 14 \times 7 + 6$.

Or September 18, 2102 is Saturday!

Further Simplify Calculation:

20 has remainder 6 when divided by 7.

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Today is day 4.

Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$

Remainder when dividing by 7? $104 = 14 \times 7 + 6$.

Or September 18, 2102 is Saturday!

Further Simplify Calculation:

20 has remainder 6 when divided by 7.

60 has remainder 4 when divided by 7.

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Today is day 4.

Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$

Remainder when dividing by 7? $104 = 14 \times 7 + 6$.

Or September 18, 2102 is Saturday!

Further Simplify Calculation:

20 has remainder 6 when divided by 7.

60 has remainder 4 when divided by 7.

Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or September 18, 2102 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$.
Or Day 6.

Years and years...

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Hmm.

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What is remainder of 365 when dividing by 7? 1

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Or September 18, 2102 is Saturday!

Further Simplify Calculation:

20 has remainder 6 when divided by 7.

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Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$.

Or Day 6. September 18, 2104 is Saturday.

Years and years...

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60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

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Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$.

Or Day 6. September 18, 2104 is Saturday.

"Reduce" at any time in calculation!

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x - y)$ is divisible by m.

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Mod 7 equivalence or *residue* classes:

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$\{\ldots, -7, 0, 7, 14, \ldots\}$

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Therefore,

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Therefore, $a + b = c + d + (k + j)m$

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$\implies a+b \equiv c+d \pmod{m}$.

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Therefore, $a+b = c+d+(k+j)m$ and since $k+j$ is integer.
$\implies a+b \equiv c+d \pmod{m}$. $\square$

Can calculate with representative in $\{0,\ldots,m-1\}$.

Notation

$x \pmod{m}$ or $\mod(x, m)$

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- remainder of x divided by m in $\{0, \ldots, m-1\}$.

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$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

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- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

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$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

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$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12$

Notation

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$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12$

Notation

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- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = 4$

Notation

$x \pmod{m}$ or $\mod(x, m)$

- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \cancel{4} = 5$

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \not{4} = 5$

Work in this system.

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \text{~~4~~} = 5$

Work in this system.

$a \equiv b \pmod{m}$.

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \cancel{4} = 5$

Work in this system.

$a \equiv b \pmod{m}$.

Says two integers a and b are equivalent modulo m.

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

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Modulus is m

Notation

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Modulus is m

$6 \equiv$

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Modulus is m

$6 \equiv 3 + 3$

Notation

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Work in this system.

$a \equiv b \pmod{m}$.

Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10$

Notation

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Work in this system.
$a \equiv b \pmod{m}$.
Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

Notation

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$6 =$

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$a \equiv b \pmod{m}$.

Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3+3 \equiv 3+10 \pmod{7}$.

$6 = 3+3$

Notation

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Modulus is m

$6 \equiv 3+3 \equiv 3+10 \pmod{7}$.

$6 = 3+3 = 3+10$

Notation

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Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

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Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

$6 = 3 + 3 = 3 + 10 \pmod{7}$.

Generally, not $6 \pmod{7} = 13 \pmod{7}$.

Notation

$x \pmod{m}$ or $\bmod(x,m)$
- remainder of x divided by m in $\{0,\ldots,m-1\}$.

$\bmod(x,m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\bmod(29,12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 =$ ~~4~~ $= 5$

Work in this system.
$a \equiv b \pmod{m}$.
Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3+3 \equiv 3+10 \pmod{7}$.

$6 = 3+3 = 3+10 \pmod{7}$.

Generally, not $6 \pmod{7} = 13 \pmod{7}$.
But probably won't take off points,

Notation

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- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

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$a \equiv b \pmod{m}$.

Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

$6 = 3 + 3 = 3 + 10 \pmod{7}$.

Generally, not $6 \pmod{7} = 13 \pmod{7}$.

But probably won't take off points, still hard for us to read.

Inverses and Factors.

Division: multiply by multiplicative inverse.

$$2x = 3 \implies (\frac{1}{2}) \cdot 2x = (\frac{1}{2}) \cdot 3 \implies x = \frac{3}{2}.$$

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Multiplicative inverse of x is y where $xy = 1$;

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Multiplicative inverse of x is y where $xy = 1$;
1 **is multiplicative identity element.**

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Poll

Mark true statements.

(A) Mutliplicative inverse of 2 mod 5 is 3 mod 5.

(B) The multiplicative inverse of $((n-1) \pmod{n}) = ((n-1) \pmod{n})$.

(C) Multiplicative inverse of 2 mod 5 is 0.5.

(D) Multiplicative inverse of $4 = -1 \pmod{5}$.

(E) $(-1)x(-1) = 1$. Woohoo.

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(C) is false. 0.5 has no meaning in arithmetic modulo 5.

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Not distinct. Common factor 2. Can't be 1.

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Proof Sketch: The set $S=\{0x,1x,\ldots,(m-1)x\}$ contains $y\equiv 1 \mod m$ if all distinct modulo m.

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For $x=4$ and $m=6$. All products of 4...

$S=\{0(4),1(4),2(4),3(4),4(4),5(4)\}=\{0,4,8,12,16,20\}$

reducing $\pmod{6}$

$S=\{0,4,2,0,4,2\}$

Not distinct. Common factor 2. Can't be 1. No inverse.

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Very different for elements with inverses.

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Bijection $\equiv$ unique pre-image and same size.

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Which is bijection?

(A) $f(x) = x$ for domain and range being $\mathbb{R}$

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Proof...: Similar.

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Therefore $d| \mod(x, y)$. And $d|y$ since it is in condition. □

Lemma 2: If $d|y$ and $d| \mod(x, y)$ then $d|y$ and $d|x$.
Proof...: Similar. Try this at home.

More divisibility

Notation: $d|x$ means "d divides x" or
$x = kd$ for some integer k.

Lemma 1: If $d|x$ and $d|y$ then $d|y$ and $d| \mod(x,y)$.

Proof:

$$\begin{aligned} \mod(x,y) &= x - \lfloor x/y \rfloor \cdot y \\ &= x - s \cdot y \quad \text{for integer } s \\ &= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\ &= (k - s\ell)d \end{aligned}$$

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Base Case: $y = 0$, "x divides y and x"
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Excursion: Value and Size.

Before discussing running time of gcd procedure...

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What is the value of 1,000,000?

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What is the value of 1,000,000?

one million or 1,000,000!

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2^{n-1} divisions! Exponential dependence on size!

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$2n$ is much faster! .. roughly 200 divisions.

Poll.

Assume $\log_2 1,000,000$ **is 20 to the nearest integer.**
Mark what's true.

Poll.

Assume $\log_2 1,000,000$ **is 20 to the nearest integer.**
Mark what's true.

(A) The size of 1,000,000 is 20 bits.
(B) The size of 1,000,000 is one million.
(C) The value of 1,000,000 is one million.
(D) The value of 1,000,000 is 20.

Poll.

Assume $\log_2 1,000,000$ **is 20 to the nearest integer.**
Mark what's true.

(A) The size of 1,000,000 is 20 bits.
(B) The size of 1,000,000 is one million.
(C) The value of 1,000,000 is one million.
(D) The value of 1,000,000 is 20.

(A) and (C).

Poll

Which are correct?

(A) gcd(700,568) = gcd (568,132)
(B) gcd(8,3) = gcd(3,2)
(C) gcd(8,3) = 1
(D) gcd(4,0) = 4

Poll

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Algorithms at work.

Trying everything

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Check 2, check 3, check 4, check 5 . . . , check $y/2$.

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Check 2, check 3, check 4, check 5 . . . , check $y/2$.

“(gcd x y)” at work.

Algorithms at work.

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"(gcd x y)" at work.

```
euclid(700,568)
```

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```
euclid(700,568)
  euclid(568, 132)
    euclid(132, 40)
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Algorithms at work.

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"(gcd x y)" at work.

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euclid(700,568)
  euclid(568, 132)
    euclid(132, 40)
      euclid(40, 12)
```

Algorithms at work.

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Notice: The first argument decreases rapidly.

Algorithms at work.

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Notice: The first argument decreases rapidly.
 At least a factor of 2 in two recursive calls.

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          euclid(4, 0)
            4
```

Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.

(The second is less than the first.)

Runtime Proof.

```
(define (euclid x y)
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    (euclid  y (mod x y))))
```

Theorem: (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

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Proof:

Fact:
First arg decreases by at least factor of two in two recursive calls.

Runtime Proof.

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After $2\log_2 x = O(n)$ recursive calls, argument x is 1 bit number.

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Runtime proof still works.

Finding an inverse?

We showed how to efficiently tell if there is an inverse.

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Extend euclid to find inverse.

Euclid's GCD algorithm.

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For x and m, if $\gcd(x,m) = 1$ then x has an inverse modulo m.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

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How do we **find** a multiplicative inverse?

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Tuesday

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Modular Arithmetic: $x \equiv y \pmod{N}$ if $x = y + kN$ for some integer k.

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