Today

Finish Euclid.

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Bijection/CRT/Isomorphism.

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Fermat's Little Theorem.

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Fermat's Little Theorem.

Quick review

Review runtime proof.

Runtime Proof.

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(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

Theorem: (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

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One more recursive call to finish.

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One more recursive call to finish.
1 division per recursive call.
$O(n)$ divisions.

□

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Case 1: $y < x/2$, first argument is y
$\implies$ true in one recursive call;

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Case 1: $y < x/2$, first argument is y
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Case 2: Will show "$y \geq x/2$" $\implies$ "$mod(x,y) \leq x/2$."

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mod (x, y) is second argument in next recursive call,
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$\lfloor \frac{x}{y} \rfloor = 1,$

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$\bmod(x,y)$ is second argument in next recursive call,
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When $y \geq x/2$, then

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$$\bmod(x,y) = x - y\lfloor \frac{x}{y} \rfloor =$$

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Poll

Mark correct answers.
Note: Mod(x,y) is the remainder of x divided by y and $y < x$.

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Note: Mod(x,y) is the remainder of x divided by y and $y < x$.

(A) $\bmod (x,y) < y$
(B) If euclid(x,y) calls euclid(u,v) calls euclid (a,b) then $a <= x/2$.
(C) euclid(x,y) calls euclid (u,v) means u = y.
(D) if $y > x/2$, $\bmod (x,y) = (x-y)$
(E) if $y > x/2$, $\bmod (x,y) < x/2$

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Finding an inverse?

We showed how to efficiently tell if there is an inverse.

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Extend euclid to find inverse.

Euclid's GCD algorithm.

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For x and m, if $\gcd(x, m) = 1$ then x has an inverse modulo m.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

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GCD algorithm used to tell **if** there is a multiplicative inverse.

How do we **find** a multiplicative inverse?

Extended GCD

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$$ax + by$$

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Check: 3(12)

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Check: $3(12) = 36$

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The multiplicative inverse of $12 \pmod{35}$ is 3.

Check: $3(12) = 36 = 1 \pmod{35}$.

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

Make *d* out of multiples of *x* and *y*..?

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gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
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        1
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How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
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```

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$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
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```

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$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

Make *d* out of multiples of *x* and *y*..?

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  gcd(12, 11)  ;;  gcd(12, 35%12)
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How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11$

Make d out of multiples of x and y..?

```
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    gcd(11, 1)  ;;  gcd(11, 12%11)
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How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12)$
Get 11 from 35 and 12 and plugin....

Make *d* out of multiples of *x* and *y*..?

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gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
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```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify.

Make d out of multiples of x and y..?

```
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  gcd(12, 11)  ;;  gcd(12, 35%12)
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How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify.

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
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How does gcd get 1 from 12 and 11?
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But we want 1 from sum of multiples of 35 and 12?

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Get 11 from 35 and 12 and plugin.... Simplify. $a = 3$ and $b = -1$.

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
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  ext-gcd(12, 11)
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Example: $a - \lfloor x/y \rfloor \cdot b = \quad 0 - \lfloor 12/11 \rfloor \cdot 1 = -1$

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Example: $a - \lfloor x/y \rfloor \cdot b =$ $1 - \lfloor 35/12 \rfloor \cdot (-1) = 3$

```
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   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

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```

Theorem: Returns (d, a, b), where $d = gcd(a, b)$ and

$$d = ax + by.$$

Correctness.

Proof: Strong Induction.[1]

[1] Assume d is $gcd(x, y)$ by previous proof.

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Base: ext-gcd($x,0$) returns ($d = x, 1, 0$) with $x = (1)x + (0)y$.

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And ext-gcd returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds!

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Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y)$

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Hand Calculation Method for Inverses.

Example: $\gcd(7,60) = 1$.

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$$
\begin{aligned}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7
\end{aligned}
$$

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Example: $\gcd(7,60) = 1$.
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$$\begin{aligned} 7(0)+60(1) &= 60 \\ 7(1)+60(0) &= 7 \\ 7(-8)+60(1) &= 4 \end{aligned}$$

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Confirm: $-119+120 = 1$

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Note: an “iterative” version of the e-gcd algorithm.

Fundamental Theorem of Arithmetic.

Thm: Every natural number can be written as the product of primes.

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Generalization: things with a "division algorithm".

One example: polynomial division.

No shared common factors, and products.

Claim: For $x, y, z \in \mathbb{Z}^+$ with $gcd(x, y) = 1$ and $x|yz$ then $x|z$.

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Observe: $x|axz$ and $x|byz$ (since $x|yz$), and x divides the sum.
$\implies x|axz + byz$
And $axz + byz = z$, thus $x|z$.

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Contradiction: $q_1 \cdot q_\ell$ and $p_1 \cdot p_k$.

No shared common factors, and products.

Claim: For $x, y, z \in \mathbb{Z}^+$ with $gcd(x, y) = 1$ and $x|yz$ then $x|z$.

Idea: x doesn't share common factors with y
so it must divide z.

Euclid: $1 = ax + by$.

Observe: $x|axz$ and $x|byz$ (since $x|yz$), and x divides the sum.
$\implies x|axz + byz$
And $axz + byz = z$, thus $x|z$.

□

Extended Euclid: computes inverses.

Extended Euclid from integer division algorithm:
or subtraction algorithm.
$\implies$ Fundamental Theorem.

Used to prove that the prime factorization of a number is unique.
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Using claim:
Conclusion: $p1 = q_i$ for some i.

Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

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Very different from elementary school: try 1, try 2, try 3...

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Internet Security: Soon.

Fundamental Theorem of Arithmetic:uniqueness

Thm: The prime factorization of n is unique up to reordering.

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If $gcd(p, q_l) = 1$, $\implies p_1 | q_1 \cdots q_{l-1}$ by Claim.
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If both prime, both only have 1 and themselves as factors.

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Proof by induction.
Base case: If $l = 1$, $p_1 \cdots p_k = q_1$.

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Induction step: From Fact: $p_1 = q_j$ for some j.
$n/p_1 = p_2 \ldots p_k$ and $n/q_j = \prod_{i \neq j} q_i$.
These two expressions are the same up to reordering by induction.
And p_1 is matched to q_j.

□

Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

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What is $x \pmod{35}$?

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What is $x \pmod{35}$?

Let's try 5.

Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

What is $x \pmod{35}$?

Let's try 5. Not $3 \pmod 5$!

Lots of Mods

$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let's try 5. Not $3 \pmod{5}$!
Let's try 3.

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If $x = 5 \pmod 7$
then x is in $\{5, 12, 19, 26, 33\}$.

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Hmmm...

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Oh, only 33 is $3 \pmod 5$.
Hmmm... only one solution.

A bit slow for large values.

Simple Chinese Remainder Theorem.

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My love is won.

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My love is won. Zero and One.

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My love is won. Zero and One. Nothing and nothing done.

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m, n)=1.

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m, n)=1.

CRT Thm: There is a unique solution $x \pmod{mn}$.

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m,n)=1.

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Proof (solution exists):

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Consider $u = n(n^{-1} \pmod{m})$.

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Let $x = au + bv$.

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Consider $v = m(m^{-1} \pmod{n})$.
$v = 1 \pmod{n}$ $v = 0 \pmod{m}$
Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$

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$u = 0 \pmod{n}$ $\quad$ $u = 1 \pmod{m}$
Consider $v = m(m^{-1} \pmod{n})$.
$v = 1 \pmod{n}$ $\quad$ $v = 0 \pmod{m}$
Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$

Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m, n)=1.

CRT Thm: There is a unique solution $x \pmod{mn}$.
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This shows there is a solution. □

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CRT:isomorphism.

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Mapping is "isomorphic":
corresponding addition (and multiplication) operations consistent with mapping.

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(A), (C), and (E)

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For a prime modulus, we can reduce exponents modulo $p-1$!

Lecture in a minute.

Extended Euclid: Find a, b where $ax + by = gcd(x, y)$.

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Fundamental Theorem of Algebra:

Unique prime factorization of any natural number.

Claim: if $p|n$ and $n = xy$, $p|x$ of $p|x$.

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Induction.

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$u = m(m^{-1} \pmod{n}) \pmod{n}$ works!

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Unique prime factorization of any natural number.

Claim: if $p|n$ and $n = xy$, $p|x$ of $p|x$.

From Extended Euclid.

Induction.

Chinese Remainder Theorem:

If $gcd(n, m) = 1$, $x = a \pmod{n}, x = b \pmod{m}$ unique sol.

Proof: Find $u = 1 \pmod{n}$, $u = 0 \pmod{m}$,

and $v = 0 \pmod{n}$, $v = 1 \pmod{m}$.

Then: $x = au + bv = a \pmod{n}$...

$u = m(m^{-1} \pmod{n}) \pmod{n}$ works!

Fermat: Prime p, $a^{p-1} = 1 \pmod{p}$.

Lecture in a minute.

Extended Euclid: Find a, b where $ax + by = gcd(x, y)$.

Idea: compute a, b recursively (euclid), or iteratively.

Inverse: $ax + by = ax = gcd(x, y) \pmod{y}$.

If $gcd(x, y) = 1$, we have $ax = 1 \pmod{y}$

$\rightarrow a = x^{-1} \pmod{y}$.

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Product of elts == for range/domain: a^{p-1} factor in range.